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THE UNIVERSITY OF ALBERTA

AN EXPERIMENTALLY DETERMINED TECHNIQUE FOR CORRESPONDENCE SCHOOL INSTRUCTION IN MATHEMATICS

A DISSERTATION
SUBMITTED TO THE GRADUATE FACULTY
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF ARTS

DEPARTMENT OF EDUCATION

BY

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EDMONTON, ALBERTA

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CHAPTER I

INTRODUCTION

The holding of the First International Conference on Correspondence School Education at Victoria, Canada, last August (1938) is concrete evidence of the growing importance of this branch of education. Delegates from Australia, New Zealand, six Canadian provinces and as many parts of the United States met for three days to exchange ideas and discuss common problems. The organization, promotion and accreditation of correspondence instruction, the preparation of courses, the work of supervisors and like questions were dealt with. The keen interest and active part taken in this conference by delegates from western Canada was due not only to the convenience of the meeting place, but also to a realization that this type of work is of special significance to our particular educational problems.

As yet, correspondence education in the west is comparatively young. British Columbia led the way in 1919 by instituting such instruction in the work of the elementary grades. Courses in secondary school subjects followed ten years later. At present, British Columbia, Saskatchewan and Manitoba have created branches of their departments of education to deal exclusively with correspondence school instruction. The Alberta department too has an interest in this work, but the control is not quite so direct because some of it is delegated to commercial houses interested in school

publications.

The chief reason for the significance of correspondence work in western Canada is the sparsity of population. In many thinly-settled areas it is not possible to provide adequate school facilities locally, so education must be obtained through other means. Then too, even where there are schools it is often difficult to offer complete courses or more advanced instruction, so correspondence courses are frequently used to supplement the regular work in the classroom. There are a number of other purposes served and needs filled by correspondence instruction that need not be mentioned here. It is sufficient to point out that progressive educational leaders are beginning to realize the importance of this work.

There are numerous factors on which the success of this branch of education depends. Among them are the administration of the work, preparation of courses, training of instructors and supervisors, and a host of minor problems. Some of these are already receiving the attention of leading educators. At the University of Nebraska, for example, much has been done toward the organization of subject matter into units particularly suited to the correspondence student. Special features are written into their courses aimed at providing motivation, supplying continuity between assignments, and linking the work to real life situations. There

is a need, however, for more extended investigation into particular phases of the work.

CHAPTER II

PURPOSE OF THE INVESTIGATION

The purpose of this paper is to suggest a technique for correspondence instruction which is considered to be specially suited to mathematics. Approximately one quarter of the Alberta course in algebra 3 has been prepared along these lines and tested out by a number of grade XII students.

Evaluation of existing courses in algebra 3

There are correspondence courses in algebra 3 being used at the present time, and they have several good features. The authorized text book is rather closely followed as to organization of work and exercises, but there are also many excellent supplementary explanations and suggestions which should be of help to the student. A number of the more difficult or significant exercises are worked out in detail, and there are also some sketches and diagrams which add vividness to the solutions. The limitations of these courses are due to their close adherence to the usual text book method of presenting theory in fairly large sections, followed by rather long sets of exercises. In spite of the fact that some text books are expressly prepared for the use of students, most teachers agree that even the best ones require additional explanations and drill exercises.

The recommended technique and its advantages

The specific aim of this thesis is to suggest that for correspondence work, the usual text book "unit of theory" be broken up, and the presentation of each part be followed by a short drill exercise aimed at testing the pupil's grasp of that particular bit of theory and fixing it in his mind. Such a procedure is known as "instructional testing", and for the unassisted student in particular, it has a number of distinct advantages. There is usually a feeling of uncertainty on the part of the pupil working alone as to whether or not he is making progress and grasping essential facts. In mathematics this is particularly true, and the best way of eliminating such a feeling is to work out problems. Consequently if there are frequent sets of short exercises, not too difficult but based definitely on the theory covered, the student gains confidence through successfully doing them. He has thus concrete evidence of progress. This arrangement also enables the pupil to locate his own difficulties, because if he has trouble with a certain exercise he knows just what element of the theory he is missing. Instructional tests are, in this sense, self-diagnostic. Long sets of exercises are uninteresting to almost all students, but specially so to correspondence students who do not receive the stimulating effect of personal help and comment. For

this reason shorter exercises interspersed with subdivided theory is more varied and interesting. Perhaps the most significant advantage is that this method resembles the usual classroom procedure. Successful teachers find that in even a thirty-five minute period some time should be taken for class exercises. Pupil interest is maintained and understanding facilitated by interspersing explanations with brief but pertinent drill problems.

Value of text book

Text books vary a good deal in their usefulness for unassisted student effort. Some are fairly well adapted to the learner's difficulties, but others are more useful as tools in the hands of a skillful teacher. Because it was felt that the latter is the case with the authorized text book in algebra 3, that course was chosen as the subject for this investigation. Five years of experience in teaching grade XII algebra has supplied abundant evidence that students have difficulty in learning from the text book alone. An inductive method, by which the pupil is led through a series of exercises to develop his own theories and formulae may be effective when there is an instructor present to prevent too much wandering from the line of argument, but when the learner must struggle on unassisted, wrong or half-correct conclusions are frequently the outcome.

CHAPTER III

MATERIALS USED

The sections dealing with differential and integral calculus, roughly one quarter of the whole course, have been used for this paper. Although the arrangement of topics, methods, and exercises from the authorized text book have been freely used, in some cases the arrangement has been distinctly altered, different methods introduced, and quite a large number of original exercises employed. These changes are an outgrowth of classroom experience, and as such, have been proven successful. In making up this set of notes and exercises, one main purpose has been kept in mind, viz. that of arranging the work in the accepted form of instructional tests. No attempt has been made to produce a set of lessons ready to be sent to a correspondence student. If such were the aim, the material would have to be divided into assignments, perhaps a few more headings added, exercises numbered, and so on. Although this additional work would not be very great, it was felt that for the purposes of this thesis, it was sufficient to illustrate the new technique.

In order to bring out more clearly the nature of instructional testing, three samples from the accompanying tests will be examined and compared with the text book and another correspondence course in algebra being used at present. The first eleven pages are taken up with the topic of gradients, and since the text book method and arrangement

have not been followed, this unit should serve as a good example of how an instructional test is made up. The aim of this part is to show the student the significance of the rate of change of one variable with respect to another, and teach him how to calculate it when one of the variables is expressed as a function of the other. Since properly-handled graphs are vivid and easily understood, this is the avenue of approach used. The topic is divided into nine parts: (1) The gradient or slope of a straight line is defined and the pupil shown how to measure it. Positive and negative slopes are distinguished. A short set of exercises follow. (2) By means of curved lines, the concept of average gradient is introduced, and the method of obtaining slopes by free hand drawing of tangents demonstrated. Brief exercises on page 25 test out these ideas. (3) Now an attempt is made to show how the slope of certain lines have special significance. This is done by discussing a temperature graph. An extension of this same graph is used as an exercise, and at this point some text book material on a similar interpretation of statistical graphs is also employed. (4) The next step is one of generalization. The graph of a certain function of x is drawn and its gradient and average gradient interpreted in theoretical terms of one variable with respect to another. A suitable exercise follows. (5) Now it is time to make the transition from graphical methods to the so-called algebraic,

but since students usually have some difficulty in connecting ordinates and abscissas with algebraic expressions, a short demonstration on this is followed by exercises of a similar kind. This is more or less in the nature of a preparatory step toward (6) the making of a formula for average gradient. The first non-graphical exercises complete this sub-division. (7) As a background for the algebraic method of determining a gradient, a short review of limits is provided. Students usually need such reviews. The short exercise made up of problems pertinent to what follows is given on page 32. (8) A final reference to graphs is made in order to connect the idea of a limit with the slope of the tangent to a curve. Some good text book material is again used at this point. (9) The gradient formula is a natural culmination, and five drill exercises in using it are provided. The above-outlined systematic arrangement of work leading by natural stages to the final goal, and testing each step in the procedure by a few exercises, is an illustration of the instructional testing technique. The text book covers this topic using three longer exercises, averaging about thirteen problems each; this scheme employs nine short exercises averaging just over four problems each.

A second example is chosen for the purpose of comparing the method of a correspondence course now in use with that of instructional testing. The topic is differentiation

by sight. The correspondence course follows the text book closely. This involves the working of two long exercises. After the first two problems have been done, the following statement is made: "no. 2 is intended to indicate that there are rules by which we can write down the differentials of certain functions of x at sight. Try to gather these rules as you solve this and the succeeding problems as it will be a great saving of time not to have to perform the steps of differentiation for each problem. There is a summary of these rules on page 67, but it would be wise not to use it except in verification of your own conclusions." This comment is followed by two or three suggestions, including a brief review--in tabular form-- of the meaning of zero, fractional and negative indices. Also: "Exercise IV (b) makes use of the rules for differentiation discovered in the previous exercise." The assignment consists of all the problems (67) from the two exercises mentioned above. The comments which have been quoted, along with the suggestions and the review of indices are good, and a student would undoubtedly receive a certain amount of direction and help from them. In contrast to this method, however, consider pages 39 to 42 of the instructional tests. The topic has been subdivided into seven parts, each with its set of drill exercises. First (1) functions such as x^n are differentiated, then (2) those of the Cx^n type. Differentiation of (3)

constants and polynomials follow, after which (4) fractional and (5) negative indices are introduced. (6) Zero and negative fractional powers come next, with a summary, and a final exercise (7) on second derivatives.

The third example is the work on the interpretation of the second derivative, and its use in distinguishing maximum from minimum values. The text book covers this topic in five problems, which are planned in such a way that when a student has worked them out, he is supposed to be capable of drawing the necessary conclusions. No comments are made or summary given. Experience has shown that for most pupils it does not work out this way. Unless a rather generous amount of help and suggestion is given by the teacher, these exercises are meaningless in so far as conclusions of any value for succeeding work are concerned. Apparently the author of the correspondence course mentioned previously feels the same way on this subject, because he has offered a short explanation and summary which are very good. Pages 44,45 and 46 of the instructional tests deal with this topic. Suitable graphs and explanations which link it up with previous work are used, and significant conclusions are set out on page 46.

CHAPTER IV

DISTRIBUTION OF THE INSTRUCTIONAL TESTS

Plans for this set of instructional tests were formulated early in the school year, and a first draft made last October. From this outline, stencils were cut and fifty copies made. In order to test their worth, these copies were distributed among various groups of grade XII pupils, teachers and correspondence students. Twenty-five sets were used by my own algebra 3 class at Grande Prairie High School, and another twelve by the students at St. Joseph's separate school in Grande Prairie. Two sets were sent to teachers in one-room high schools, another set to a student attempting to study the course with no assistance other than the text book, and still another to a teacher who wanted to become familiar with the calculus portion of the "new" algebra 3 course before registering in a mathematics course at the University of Alberta. Contact was made with the Western Canada Institute of Calgary, publishers of correspondence courses, with the result that their instructor in algebra 3, Mr. T. E. A. Stanley of Calgary, expressed an interest in the project and offered to give his opinion of the tests. A set was sent to him, and also to each of two of his regular correspondence students, whose names he suggested. On the basis of results, comments and suggestions received from these sources, certain revisions were made and the tests prepared in their final form.

CHAPTER V

REVISIONS

With my own algebra 3 class, the same methods as those outlined in the tests would have been used in any case, so having a set of notes and exercises for each student merely facilitated the regular work. In order, however, to give the tests a fair try-out, daily assignments were made without preliminary discussions or explanations. This work lasted for approximately two months. To illustrate the method used, consider the last half of page 28 and page 29 of the tests. The class was told to study the material and work the exercises from this section. On the following day, the correct answers were supplied and a few brief questions asked to determine whether or not the essential facts had been grasped. In this particular instance, most of the class had all the problems worked correctly, everyone seemed to have mastered the idea of linking abscissas and ordinates with certain numbers or algebraic expressions, and the few errors made in exercise were chiefly mechanical ones. It should hardly be necessary to give here a detailed summary of the errors made throughout the tests. There were, however, three places where the results seemed to warrant making slight revisions: (1) On page 38, the original exercises said nothing about the "limit method" of differentiating, with the result that when this expression was used later, it was not meaningful to all the students. Consequently, in the revised form, a

short paragraph (as at the top of page 38) was inserted to introduce this term. (2) The explanation of the second derivative (pages 44, 45 and 46), as contained in the original tests, seemed to cause some difficulty, so in the revision an attempt was made to clarify this part. Instead of discussing "the graph of $y =$ some function of x", estimating and tabulating its slope at four places (including a maximum turning point), and drawing a second graph showing "the relationship between $\frac{dy}{dx}$ and x", a definite function $(33 + 10x - x^2)$ was chosen, corresponding values for $\frac{dy}{dx}$ and x worked out and put in table form, and another graph drawn based on this table, i.e. a graph of $\frac{dy}{dx} = 10 - 2x$". It was felt that the discussion of a specific case would be more readily understood. Generalized statements follow in the conclusion. (3) The exercises which gave the class most trouble were those on quadratic functions (middle of page 48) and the two succeeding problems on other special types of functions. An attempt had been made in formulating these problems to provide a few "thought questions" which would really test out the pupil's understanding of the conclusions from page 46. The exercise on quadratic functions was originally worded: "(1) For the quadratic function:

$y = ax^2 + bx + c$, $\frac{dy}{dx} = 2ax + b$ and $\frac{d^2y}{dx^2} = 2a$

(a) Why can a quadratic function have only one turning point?

(b) How can you tell from the function whether this point is a maximum or a minimum?" Although difficulty with this problem was general throughout the class, as soon as a particular quadratic function was presented, both (a) and (b) were correctly answered, so apparently the trouble was not with the interpretation of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ but with "the quadratic function $y = ax^2 + bx + c$". In re-wording this problem, it will be noticed that $y = ax^2 + bx + c$ was presented as a general form, with a specific example also given. Parts (a) and (b) were made a little more directive and (c) added to provide a brief drill on (b). Similar changes were made in problems (2) and (3) (page 48).

Taken as a whole, the reaction of the class was very good, and progress quite satisfactory. The students seemed to find the tests useful as an outline and summary of the work covered. The results of a short test given after completing the assignments showed an intelligent grasp of all topics. Opinions of the class members indicated that the work had been varied and interesting.

CHAPTER VI

RESULTS AND CRITICISMS

St. Joseph's separate school at Grande Prairie has one high school room where grades 10, 11 and 12 are taught. The class in algebra 3 this year is rather larger than usual and the Sister in charge was pleased to accept sets of the instructional tests for members of her class. The time available for the subject in a school of this type is, of course, much less than in a departmentalized high school. Consequently it was felt that this was a good place for trying out the tests. Although I have received no written report from the teacher, her opinions as expressed in a recent personal interview, indicate that the tests were found very useful. They completed the entire set some time ago, and no significant revisions were suggested.

From correspondence school instructor

As mentioned above, a set of tests was sent to Mr. T. E. A. Stanley of Calgary. His interest in the project was greatly appreciated, because he is not only an instructor in an algebra 3 correspondence course, but also a mathematician of recognized ability. Most cordial comments and suggestions were received from him. He says, in part: "My opinion of the treatment is that these exercises would be of great benefit to a correspondence student. The work on gradients seems to

be particularly good. One difficulty with the course as at present used is that wholesale assignments are made that almost invariably smother and discourage the student. Then the text book doesn't help much. The student is left to discover a hidden line of argument that he seldom recognizes until he has first got a line on it from some other source. Your treatment commends itself to me..... The present treatment will I am afraid squeeze out a lot of those attempting to take the course by correspondence."

Mr. Stanley suggested that copies of the tests be sent to two of his regular correspondence students whose names and addresses he supplied. This was done, and a response received from both of them. They were asked to offer any comments or criticisms, and invited to send difficulties to me. Although these sets were sent out early in December, neither student has yet asked for assistance with any troubles. Judging from correspondence received from them, however, they are finding the tests interesting and useful.

From correspondence student

Very satisfactory results were received from a student at Fort Kent (Alberta) who is attempting to study algebra 3 this year without the help of even a correspondence course or manual. A set of tests was sent to this student with the request that all exercises be returned to me for checking and comments. The first group of problems--about one third

of the total--were received early in January, and were exceptionally well done. Very few errors had been made, and an accompanying letter says: "I may truly say that I liked the exercises." Three weeks later all the remaining problems arrived, and they too were done remarkably well. Other than a few mechanical errors, the only difficulty experienced by this student was in solving the problems on the quadratic and cubic functions already referred to. This was the only place where she seemed to lack the complete command of the work, so evident in all her other exercises. In the letter received with the second set of problems, she says: "Your course is a great help, your notes are so clearly stated that we cannot help but understand. I would gladly recommend it to any students taking algebra 3 privately."

The teacher to whom a set of tests was sent studied the old algebra 3 course before the work on calculus was introduced, so he had a better general background in mathematics than the average student would have, but was anxious to do some introductory work in calculus before starting his university mathematics. He was not asked to send in all the exercises, but invited to make comments or suggestions. In a communication recently received from him, he says: "Although I have had no previous work in differentiation and integration I have had no difficulty in understanding the material as presented in your outline, without the assistance

of a teacher. I think that there are two reasons why your presentation of this work is simpler to master than the same work in the text book.

First, you have not followed the usual practice in texts of building up too much theory before the student is given an opportunity to practice what has been learned. When several points have been presented and must be put to use in an exercise the student is confused by having to think of all of them to-gether. Instead, your method seems to be to divide the material up into smaller portions, which gives the student an opportunity to practice each point learned without too many facts to worry about while doing so. A good example of this difference is seen in the presentation of the formulas for the short-cuts in differentiation. Where the text develops these all together, you have stressed one at a time with exercises which give practice in each one as learned. It seems to me that while several such formulas to learn at once becomes quite a task, it becomes comparatively simple when they are taken one at a time.

The second reason why I found your outline helpful was that there are frequent summaries and reviews which aid the student in deciding what points to stress in learning the material. This is especially helpful to the student working alone, because when considerable work has been covered it is difficult to know just what to emphasize and the student

often tries to memorize an unnecessary amount of detail for fear of missing some important fact.

In addition to these points I may say that I found the work very clearly presented and the examples fully explained. I have found the work as presented in this outline very interesting and pleasant."

CHAPTER VII

CONCLUSIONS AND SUGGESTED EXTENSIONS

The results of this experiment substantiate the advantages claimed for instructional tests. Systematic arrangement and variety provide motivation, the short exercises are more or less self-diagnostic, and the smaller units of theory give the student greater confidence. If further evidence were desired, it would be interesting to prepare the entire course along these lines and let some regular correspondence students work through it. I have every confidence that in such a situation, any pupil of average ability could be successful on the final examination set by the department of education. The teacher at St. Joseph's separate school expressed the wish that similar tests were available for the other topics in the course, and one of the correspondence students suggested by Mr. Stanley wrote recently to inquire whether or not more such material could be supplied. It would be interesting to try this method on other high school courses. Whether or not it would be successful when applied to sciences, languages, or other subjects, is of course problematical. The results of this investigation however, provide conclusive evidence that the technique of instructional testing is suitable for correspondence school instruction in mathematics.

CHAPTER VIII

INSTRUCTIONAL TEST USED IN THE INVESTIGATION

Gradients

The slope or gradient of a line is measured by the amount it rises in a given horizontal distance. For example, the slope of the line AB in fig.1 is the rise BC (3 cm.) divided by the horizontal distance AC (5 cm.), or $\frac{3}{5}$. It is often expressed as a "slope of 3 in 5", or as a "60% gradient."

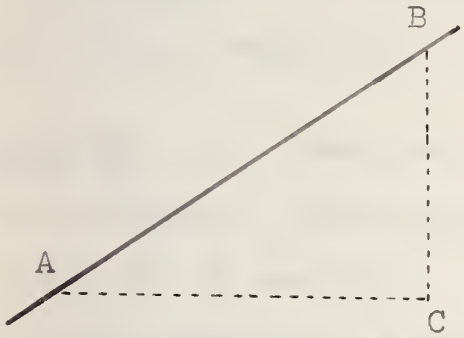


Fig. 1

REMEMBER: The slope of a line is expressed as a fraction--

$$\frac{\text{vertical rise}}{\text{corresponding horizontal distance}}$$

When dealing with these horizontal and vertical distances, it is customary, as in graphical work, to consider measurements to the right or upward as positive, and those to the left or downward as negative. Thus in fig.1, if measurements are made from A to B via C, it is evident that AC is +5 cm., and CB +3 cm., making the slope $+\frac{3}{5}$. According to this scheme, some lines have negative slopes: for example in fig.2, measuring from D to E via F, DF =

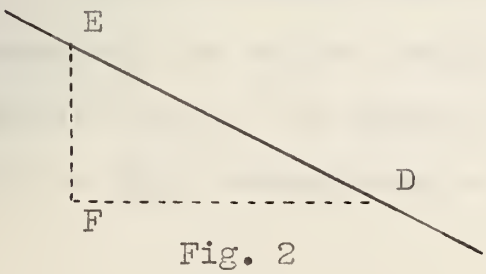


Fig. 2

-4 cm. and FE = +2 cm., making the slope of the line DE $\frac{+2}{-4}$ or $-\frac{1}{2}$. (Note: the same result might have been obtained by measuring from E to D. EF = -2, FD = +4, slope: $\frac{-2}{+4}$ or $-\frac{1}{2}$.)

<u>IT IS EVIDENT</u>: Lines which rise to the right have positive slope; those which fall to the right have negative slope.

<u>EXERCISES</u>: (1) From the text book: page 57, no. 3.

(2) In figure 3, what is the slope of each of the lines: OA, OB, PC, PD and RT ?

(3) In the middle of a sheet of graph paper, mark a point P, and draw the following lines:

PA	having	a	slope	of	$+\frac{2}{3}$.
PB	"	"	"	"	$-\frac{7}{5}$.
PC	"	"	"	"	+4.
PD	"	"	"	"	0.

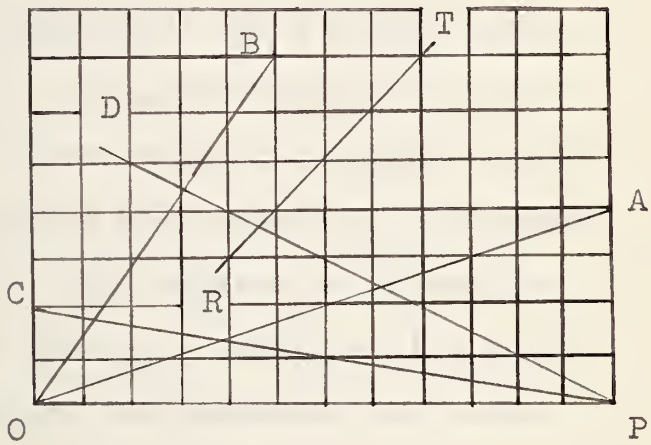


Fig. 3

(4) On graph paper, plot the following points: A (4, 5), B (-5, 3), C (-3, -1), D (2, -7). Calculate the slope of the lines: AB, BC, CD, DA, CA, and BD.

It is evident that a straight line has the same slope throughout its entire length, but this is not the case with a curved line. For example, the gradient of the curved line ABCD in fig.4 varies from a steep slope at A to a gradual

slope at D. In this case the <u>average slope</u> between two points, such as A and C, may be determined by finding the slope of the line joining A and C. This is $+\frac{3}{2\frac{1}{2}}$ or $+\frac{6}{5}$. Similarly, the <u>average slope</u> between B and D is $+\frac{2}{5}$.

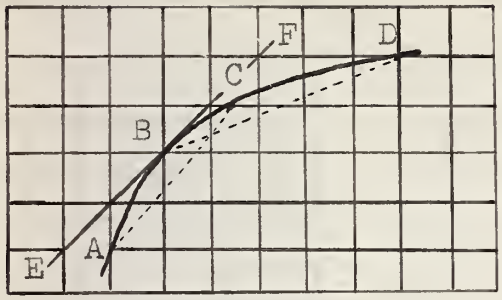


Fig. 4

The slope at a particular point on such a curve is more difficult to determine accurately, but a fairly close approximation may be obtained. It is not difficult to imagine that if the slope of a curve remained constant beyond B, the line would be straight from that point on, and would go to F instead of to C and D. This suggests a method for obtaining the slope at B: by drawing a straight line just touching the curve at that point, and measuring its slope. In fig.4, EF is such a line (called a <u>tangent</u>) and by using any two convenient points on it, its slope may be shown to be +1. Similarly the slope of this curve at any other point may be found by drawing a tangent at the point and measuring its slope. Of course when these tangents are drawn by eye, the results are only approximate.

<u>EXERCISES</u>: (1) In each of the figures 5, 6, and 7, find the average gradient of the curve between the points A and C.

(2) By drawing tangents as accurately as possible, find in each figure the slope of the curve at the point B.

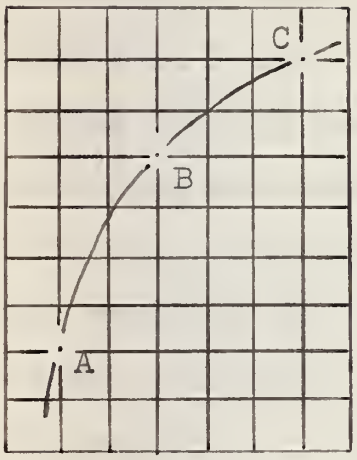


Fig. 5

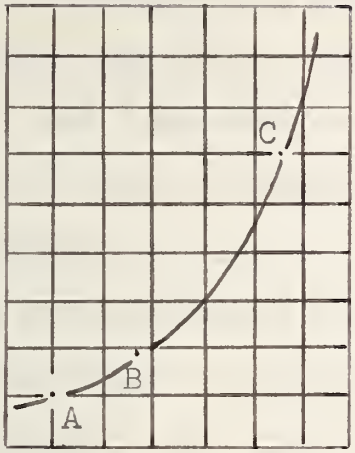


Fig. 6

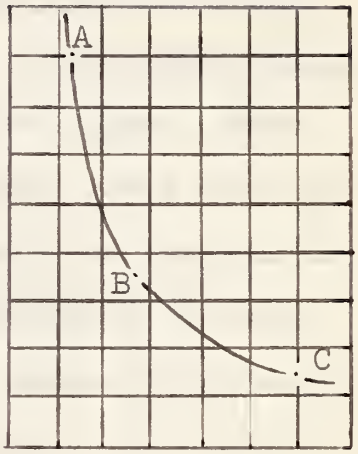


Fig. 7

Figure 8 is a graph based on observed temperature changes over an 8-hour period. Observe that between 8 a.m. and 9 a.m. the temperature rose 10°, and between 9 and 10 it also rose 10°. If it had continued rising at this rate, the graph would have been a straight line. Between 11 and 12 noon, however, the rise was only 5° and between 12 and 1 p.m. only $2\frac{1}{2}$°. This decreasing rate is shown by the more gradual slope of the graph. Appar-

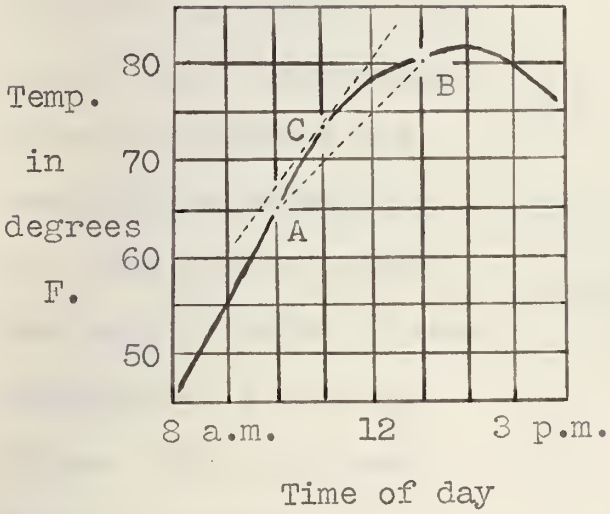


Fig. 8

ently the gradient of this graph indicates the rate at which the temperature was changing. Between 10 a.m. and 1 p.m. the temperature rose 15°. Therefore the <u>average rate of change</u> over this 3-hour interval was 5° per hour. The <u>average slope</u> of the graph between A and B determines this. (<u>Note</u>: When measuring the slope of a line in a statistical graph such as this one--or in any graph--counting the number of squares in the vertical rise and the corresponding horizontal distance is not sufficient. The number of units in these distances is determined by the temperature and time scales.) By drawing a tangent at the point C, it can be estimated that at 11 a.m. the temperature was rising at the rate of approximately 6° per hour.

<u>EXERCISES</u>: (1) Figure 9 shows a continuation of the same graph as in fig.8. From fig.9, find the average rate of change of the temperature between 4 p.m. and 8 p.m., also the rate at which it was changing at 4 p.m. and 8 p.m. (Use the tangents as shown.)

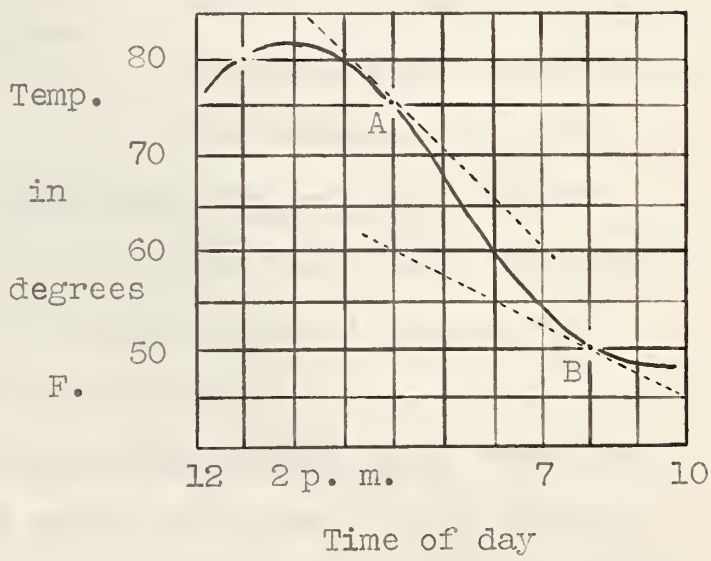


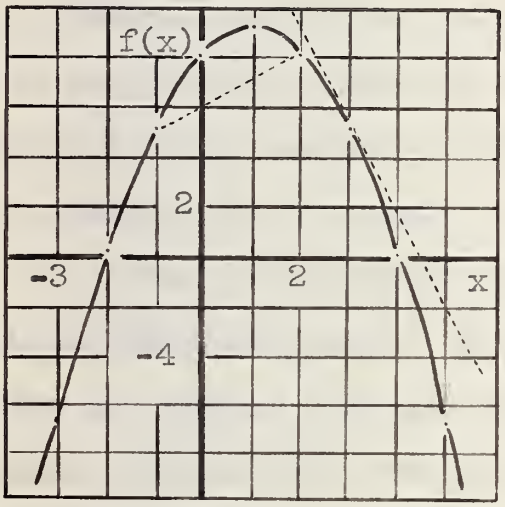
Fig. 9

(2) It is suggested that the student study example

III on pages 50-51 from the text book. Then problems number 1, 2, and 3 from page 52 should be attempted.

(3) From the text book: page 53, number 5.

In the temperature graph discussed above, the slope indicated the rate of change of temperature, or the rate at which one variable (temp.) was changing with respect to the other (time). Fig.10 is a graph showing the relationship between two variables designated by x and f(x). The slope of this graph therefore, indicates the rate at which the function of x is changing with respect to x.



$f(x) = 8 + 2x - x^2$

x	-3	-2	-1	0	1	2	3	4	5
f(x)	-7	0	5	8	9	8	5	0	-7

Fig. 10

For example: (1) The <u>average rate of change of f(x) with respect to x</u> between $x = -1$ and $x = +2$ is $\frac{+3}{+3}$ or $+1$. This means that throughout this particular range, f(x) is increasing at the average rate of 1 unit for each unit increase in x.

(2) The <u>rate of change of f(x) with respect to x</u> when $x = 3$ is approximately -4. This means that when $x = 3$, f(x) is decreasing at the rate of 4 units for each unit increase in x.

<u>EXERCISES</u>: Draw the graph of the function $x^3 - 4x^2 - 11x + 30$ between $x = -4$ and $x = +6$. (Note: after making a table of values over this range plan scales so that the graph will take up the major portion of a sheet of graph paper; an accurately-drawn graph with "smooth" curves is essential for this problem.)

(1) From the graph find the average rate of change of $f(x)$ with respect to x over the interval $x = -1$ to $x = +3$.

(2) By drawing tangents, estimate the rate of change of $f(x)$ with respect to x when $x = -2, 0, +4$.

Thus far gradients and average gradients have been determined graphically. Although graphical methods such as these are vivid, they are lacking in accuracy. This is of course particularly true of the method used above for finding the slope of a curve by drawing a tangent by eye. There are "algebraic" methods of obtaining both average gradients and gradients, which yield accurate results. The one for getting the average gradient will be considered first.

Fig.11 on the next page shows a portion of the graph of $f(x) = x^2 + 1$. If the x co-ordinate (i.e. the abscissa) of any point on this graph is known, the $f(x)$ co-ordinate (i.e. the ordinate) may be determined by using $f(x) = x^2 + 1$. For example, if the abscissa of P is 2, the ordinate must be $f(2) = 5$. If the abscissa of Q is r, the ordinate is $f(r) =$

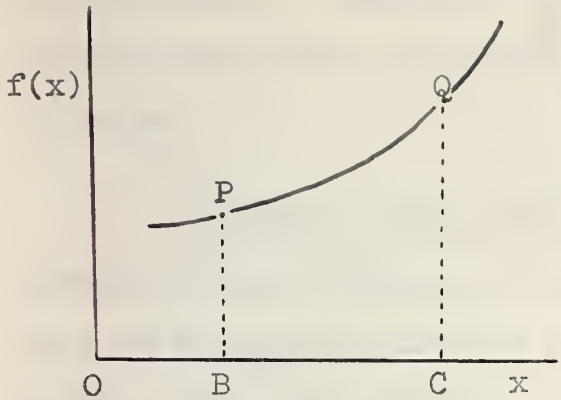


Fig. 11

$r^2 + 1$. It is important to notice that these co-ordinates represent distances: OB = 2 units, PB = 5 units, OC = r units, QC = $r^2 + 1$ units. These "units" are, of course, determined by the scales along the x and f(x) axes.

<u>EXERCISES</u>: (1) Fig.12 shows the graph of $f(x) = 2x^2 + 3$. If the abscissa of the point P is 3, and the line BC is e units long, find: the ordinate of P, the co-ordinates of Q, and the lengths of OC, PB, QC, QA, and PA.

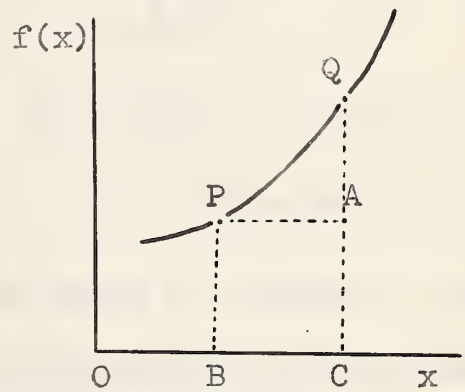


Fig. 12

(2) Fig.13 shows the graph of $f(x) = \frac{1}{x}$. If OB = $\frac{1}{2}$ and BC = g, find: the co-ordinates of P and Q, and the lengths of OC, PB, QC, PA and AQ.

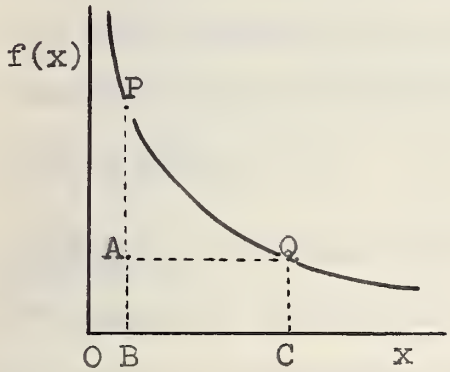


Fig. 13

(3) Suppose fig.12 shows the graph of $f(x) = x^2 + 2$. If OB = k, and BC = n, find the lengths of OC, PB, QC, PA and QA.

(4) In each of questions (1), (2) and (3), using the measurements found, write down the expression for the slope of the line PQ.

A FORMULA FOR AVERAGE GRADIENT: Suppose fig.14 is the graph of some function of x, and OB = x units and BC = h units. Then PB = f(x), PA = h, QC = f(x + h), and QA = f(x + h) - f(x). It follows that the average gradient between P and Q is $\frac{QA}{PA}$ or $\frac{f(x + h) - f(x)}{h}$.

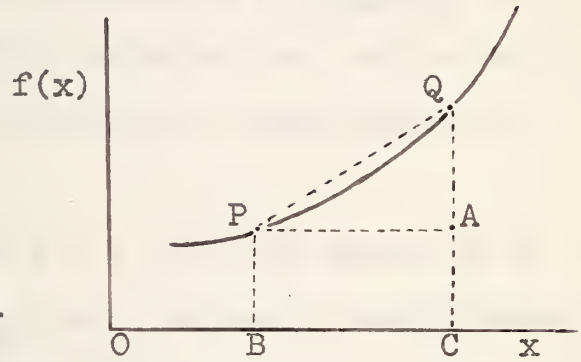


Fig. 14

This is an important formula which makes it possible to find the average rate of change of any function of x with respect to x (over a given interval) without using a graph. For example, to find the average change of the function $x^2 + x - 1$ per unit increase in x, over the range from $x = 2$ to $x = 5$, substitute in the above formula, $x = 2$ and $h = 3$:

$$\frac{f(5) - f(2)}{3} = + 8$$

EXERCISES: By using the above formula, find the average rate of change of each of the following functions with respect to x, over the intervals stated:

(1) $f(x) = x^2 + 2x - 3$ (from $x = 1$ to $x = 3$)

(2) $f(x) = 3x^2 + 7x - 2$ ($-2 \leqq x \leqq -1$)

(3) $f(x) = 5 - x - 2x^2$ $\quad (0 \leqq x \leqq -2)$

(4) $f(x) = \sqrt{x}$ $\quad (+4 \leqq x \leqq +9)$

(5) $f(x) = \frac{1}{x}$ $\quad (+\frac{1}{2} \leqq x \leqq +2)$

(6) Review exercise: text book--p.57, no 7.

The algebraic method of obtaining a <u>gradient</u> is a very important part of grade XII algebra, but before this method is determined, it is advisable to review some of the work on limits.

Notice that $\frac{x^2 + x}{x} = x + 1$ for all values of x <u>except 0</u>. When $x = 0$, $\frac{x^2 + x}{x} = \frac{0}{0}$, which is meaningless, but $x + 1 = 1$. To get the table of values shown below, the form $x + 1$ may be used providing x is not equal to zero:

$$f(x) = \frac{x^2 + x}{x} = x + 1$$

x	0.5	0.1	0.01	0.0001	0.000001
f(x)	1.5	1.1	1.01	1.0001	1.000001

It is evident from this table that as x is made smaller, f(x) approaches the value + 1. This value is called the limit of $\frac{x^2 + x}{x}$, and is usually written thus:

$\underset{x \to 0}{\text{Lt}} \frac{x^2 + x}{x} = +1$. This statement is read: "the limit of $\frac{x^2 + x}{x}$ when x approaches 0 is + 1." It is not possible to let $x = 0$, because then $\frac{x^2 + x}{x} = \frac{0}{0}$.

Similarly: $\underset{h \to 0}{\text{Lt}} \dfrac{(1 - h)^2 - 1}{h} = \underset{h \to 0}{\text{Lt}} \dfrac{1 - 2h + h^2 - 1}{h}$

$$= \underset{h \to 0}{\text{Lt}} \; -2 + h$$

$$= -2 \text{ (providing } h \text{ is not } = 0)$$

<u>EXERCISES</u>: Find the following limits:

(1) $\underset{h \to 0}{\text{Lt}} \dfrac{2(2 + h)^2 - 8}{h}$

(2) $\underset{r \to 0}{\text{Lt}} \dfrac{(1 + r)^2 - 3(1 + r) + 2}{r}$

(3) $\underset{e \to 0}{\text{Lt}} \dfrac{\frac{3}{5 + e} - \frac{3}{5}}{e}$

(4) If $f(x) = x^3$ evaluate: $\underset{h \to 0}{\text{Lt}} \dfrac{f(2 + h) - f(2)}{h}$

(5) If $f(x) = 2x^2 - x - 3$,

evaluate: $\underset{h \to 0}{\text{Lt}} \dfrac{f(a + h) - f(a)}{h}$

Figure 15 shows the graph of $f(x) = x^2 + 1$. If the co-ordinates of P are (1, 2) and BC = h, then the co-ordinates of Q are $[(1 + h), f(1 + h)]$. Therefore, the slope of PQ is $\frac{QA}{PA}$ or

$$\frac{f(1 + h) - f(1)}{h} = 2 + h$$

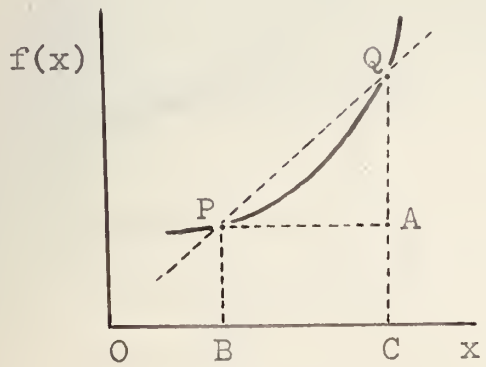
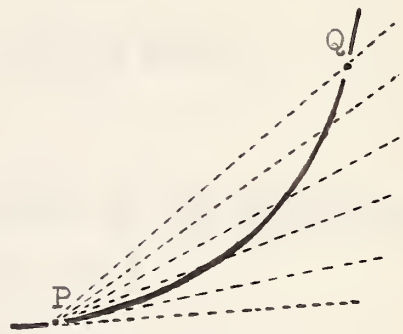


Fig. 15

Now the interval h as shown is fairly long. Suppose that with P and B remaining fixed, h is gradually shortened. C and

A will move closer to B and P, and Q will move down the curve toward P. Figure 16 shows how the slope of PQ changes when h is shortened: as Q gets closer to P, the line PQ approaches the position of tangent to the curve at P. Therefore, when $h \to 0$, the slope of PQ $\to$ slope of tangent at P. But the slope of PQ $= 2 + h$, and $\underset{h \to 0}{\text{Lt}} \; (2 + h) = 2$.



Therefore: slope of tangent at P = 2.

Fig. 16

<u>EXERCISES</u>: From the text book: pp. 49-50, nos. 6, 7, 8. It is suggested that the student study the example shown on pp. 54-56 of the text book and then attempt problems from pp. 57-58, nos. 5, 6, 8, 9.

<u>A GRADIENT FORMULA</u>: Suppose fig.17 is the graph of some function of x, and OB = x units and BC = h units.

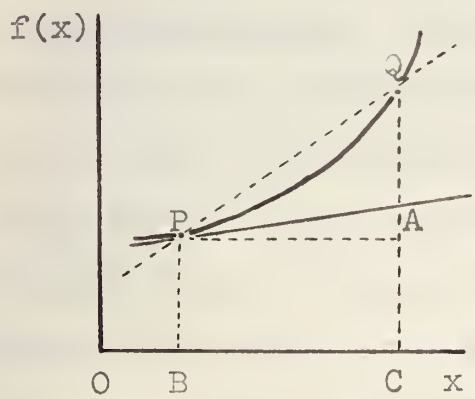


Fig. 17

Then the average slope between P and Q is $\dfrac{f(x + h) - f(x)}{h}$.

It follows that <u>the gradient at P</u> is: $\underset{h \to 0}{\text{Lt}} \; \dfrac{f(x + h) - f(x)}{h}$.

This is the formula for finding the <u>rate of change of any function of x with respect to x</u>, at any specified value of x.

For example: to find the change in the function $x^2 + x - 1$ per unit increase in x, at $x = 3$, substitute $x = 3$ in the above formula: $\underset{h \to 0}{\mathrm{Lt}} \dfrac{f(3 + h) - f(3)}{h} = \underset{h \to 0}{\mathrm{Lt}} (7 + h) = + 7$

(The summary on page 61 of the text book is good).

EXERCISES:

Find the rate of change of f(x) in each of the following:

(1) $f(x) = 2x^2 - 3x + 1$, at $x = 2$

(2) $f(x) = 7 - x^2$, at $x = 3$

(3) $f(x) = \dfrac{1}{x^2}$, at $x = 1$

(4) $f(x) = x^3 + x + 1$, at $x = a$

(5) $f(x) = x^2 + 5x - 9$, at $x = x$

Differentiation

The process of finding the rate of change of a function of x with respect to x, by the algebraic method is called differentiation. Since it is customary to use various symbols in the formulae, the student should be familiar with all of them. For example, $\triangle x$ is frequently used in place of h, and y in place of f(x). ($\triangle$ is a Greek letter, pronounced "delta", so $\triangle x$ is called "delta x"). This does not change the meaning of the formulae, but it does alter their appearance. Figures 18, 19, and 20 show the same graph:

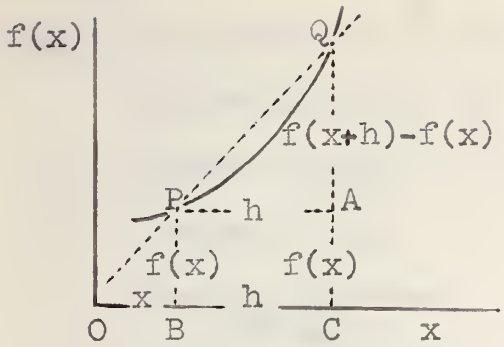


Fig. 18

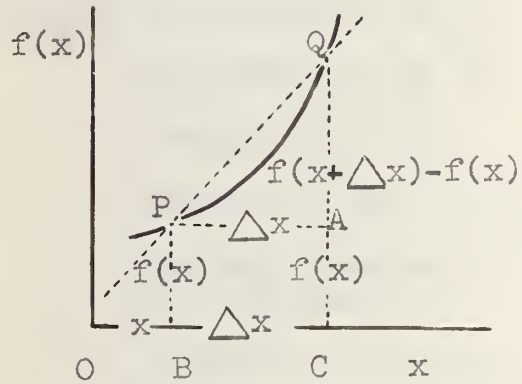


Fig. 19

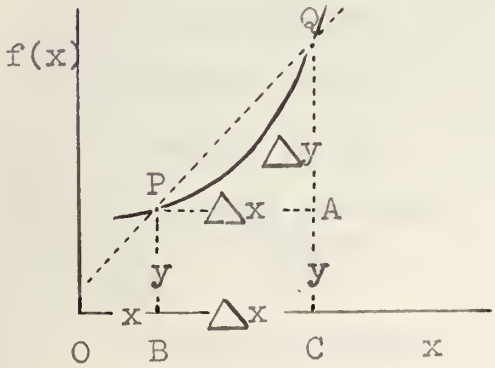


Fig. 20

In fig.18 the old symbols are used; in fig.19 h has been replaced by $\triangle x$; in fig.20 both h and f(x) have been replaced by $\triangle x$ and y. Since $\triangle x$ is a special symbol meaning "a change in x", the corresponding change in f(x) could be designated by $\triangle f(x)$, but is usually given in the form

$$f(x + \triangle x) - f(x),$$

which means the same: i.e. new value of the function minus old value. When y is used in place of f(x) however, the symbol $\triangle$ is used for both x and y: $\triangle x$ as a change in x (BC) and $\triangle y$ as a change in y (QA).

SUMMARY:

Fig.18	Co-ordinates of P:	$x,\ f(x)$
	Co-ordinates of Q:	$(x + h),\ f(x + h)$
	Length of PA:	h
	Length of QA:	$f(x + h) - f(x)$
	Average gradient formula:	$\frac{f(x + h) - f(x)}{h}$
	Gradient formula:	$\underset{h \to 0}{\mathrm{Lt}} \frac{f(x + h) - f(x)}{h}$
Fig.19	Co-ordinates of P:	$x,\ f(x)$
	Co-ordinates of Q:	$(x + \triangle x),\ f(x + \triangle x)$
	Length of PA:	$\triangle x$
	Length of QA:	$f(x + \triangle x) - f(x)$
	Average gradient formula:	$\frac{f(x + \triangle x) - f(x)}{\triangle x}$
	Gradient formula:	$\underset{\triangle x \to 0}{\mathrm{Lt}} \frac{f(x + \triangle x) - f(x)}{\triangle x}$
Fig.20	Co-ordinates of P:	$x,\ y$
	Co-ordinates of Q:	$(x + \triangle x),\ (y + \triangle y)$
	Length of PA:	$\triangle x$
	Length of QA:	$\triangle y$
	Average gradient formula:	$\frac{\triangle y}{\triangle x}$
	Gradient formula:	$\underset{\triangle x \to 0}{\mathrm{Lt}} \frac{\triangle y}{\triangle x}$

The most commonly-used formulae are:

$\dfrac{f(x + \triangle x) - f(x)}{\triangle x}$ for finding average gradient and

$\underset{\triangle x \to 0}{\mathrm{Lt}} \dfrac{f(x + \triangle x) - f(x)}{\triangle x}$ for getting the gradient.

$\underset{\triangle x \to 0}{\mathrm{Lt}} \dfrac{f(x + \triangle x) - f(x)}{\triangle x}$ is usually abbreviated to $\dfrac{d}{dx} f(x)$ or $\dfrac{df(x)}{dx}$, and is called "the differential coefficient of f(x) with respect to x", or the "the derived function of f(x)", or "the derivative of f(x)". When y replaces f(x), $\underset{\triangle x \to 0}{\mathrm{Lt}} \dfrac{\triangle y}{\triangle x}$ is abbreviated to $\dfrac{d}{dx} y$ or $\dfrac{dy}{dx}$, and is called "the differential coefficient of y with respect to x", or "the derived function", or "the derivative". (NOTE: $\dfrac{dy}{dx}$ does not mean d times y divided by d times x; the whole fraction is a unit--a symbol.)

<u>EXAMPLES</u>: Differentiate with respect to x:

(1) $x^2 + 1$, (2) $y = 3 - \frac{1}{x}$.

(1) $\dfrac{d}{dx} f(x)$ or $\dfrac{d}{dx}(x^2 + 1)$

$$= \underset{\triangle x \to 0}{\mathrm{Lt}} \frac{\left[(x + \triangle x)^2 + 1\right] - \left[x^2 + 1\right]}{\triangle x}$$

$$= \underset{\triangle x \to 0}{\mathrm{Lt}} \frac{2x.\triangle x + \triangle x^2}{\triangle x}$$

$$= \underset{\triangle x \to 0}{\mathrm{Lt}} (2x + \triangle x)$$

$$= 2x$$

$$(2)\ \frac{dy}{dx} = \underset{\triangle x \to 0}{\mathrm{Lt}} \frac{\left(3 - \frac{1}{x + \triangle x}\right) - \left(3 - \frac{1}{x}\right)}{}$$

$$= \underset{\triangle x \to 0}{\mathrm{Lt}} \frac{-x + x + \triangle x}{\triangle x . x(x + \triangle x)}$$

$$= \underset{\triangle x \to 0}{\mathrm{Lt}} \frac{1}{x^2 + \triangle x . x}$$

$$= \frac{1}{x^2}$$

Since the final step in both of these examples is obtaining the limit of an expression as $\triangle x \to 0$, this is commonly called the "limit method" of getting the differential coefficient.

From the second example, $\frac{dy}{dx} = \frac{1}{x^2}$. This statement is significant both graphically and algebraically: (1) It gives the gradient of the graph of $y = 3 - \frac{1}{x}$ at any point, e.g. when $x = 2$, the slope of the curve is $\frac{1}{4}$. (2) It gives the rate of change of y with respect to x for any value of x, e.g. when $x = 3$, y is increasing $\frac{1}{9}$ of a unit for each unit increase in x.

The student is advised to study the text book: pp. 62-64.

<u>EXERCISES</u>: Differentiate with respect to x, using the limit method: (1) $5x^2 - 3x + 2$ (2) $2x + \frac{1}{x^2}$

(3) $y = 4 + x - 3x^2$ (4) $y = \frac{1}{2x} - x^2$

From the derivatives obtained in nos. 1 and 3, find the slope of the graphs when $x = 2$; from the derivatives obtained in

nos. 2 and 4, find the rate of change of the f(x)--or y-- with respect to x when $x = 2$.

REVIEW AND SUMMARY: Equivalent forms for the derivative are--

$$\underset{\triangle x \to 0}{\mathrm{Lt}} \frac{f(x + \triangle x) - f(x)}{\triangle x} \; ; \quad \underset{\triangle x \to 0}{\mathrm{Lt}} \frac{\triangle y}{\triangle x} \; ; \quad \frac{d}{dx} f(x) \; ;$$

$$\frac{d}{dx} y \; ; \quad \frac{df(x)}{dx} \; ; \quad \frac{dy}{dx}.$$

Each of these six forms may be called: (1) differential coefficient of f(x)--or y-- with respect to x, (2) derived function of f(x), (3) derivative. Each may be used for determining: (1) the gradient of the graph of a f(x), (2) the rate of change of a function with respect to x.

Differentiation by the limit method is in some cases a long involved process, but there are certain short cuts which can be used.

Examine the following functions and their derivatives:

$y = x^3$, $\frac{dy}{dx} = 3x^2$; $y = x^5$, $\frac{dy}{dx} = 5x^4$; $y = x^{10}$, $\frac{dy}{dx} = 10x^9$.

From these examples, it is evident that if $y = x^n$,

$\frac{dy}{dx} = n.x^{n-1}$ (where n is any number).

This is an important "short cut" differentiation formula.

EXERCISES: Using the short method, write down the derivatives for: (1) $y = x^8$ (2) $y = x^4$ (3) $y = x^2$ (4) $y = x^9$

Text book: pp. 68-69, nos. 1, 14.

Examine the following functions and their derivatives:

$y = 5x^3$, $\frac{dy}{dx} = 15x^2$; $y = 2x^5$, $\frac{dy}{dx} = 10x^4$; $y = \frac{1}{2}x^{10}$, $\frac{dy}{dx} = 5x^9$.

The formula covering these cases is: if $y = C.x^n$, $\frac{dy}{dx} = C.n.x^{n-1}$ (where C and n are any numbers).

EXERCISES:

Using the short method, differentiate with respect to x:

(1) $y = 3x^7$ (2) $y = \frac{1}{8}x^4$ (3) $y = 2x$

(4) $y = \frac{x^6}{2}$ (5) $y = \frac{4x^3}{9}$

Fig.21 shows the graphs of $y = 1$, $y = 4$, and $y = 6$. These lines all have zero slope. We conclude, therefore, that if $y = C$, $\frac{dy}{dx} = 0$ (where C is any number). Note also, that in differentiating an expression of more than one term, each term is differentiated independently, for example: $\frac{d}{dx}(3x^4 - \frac{x^2}{2} + 5) = 12x^3 - x + 0$.

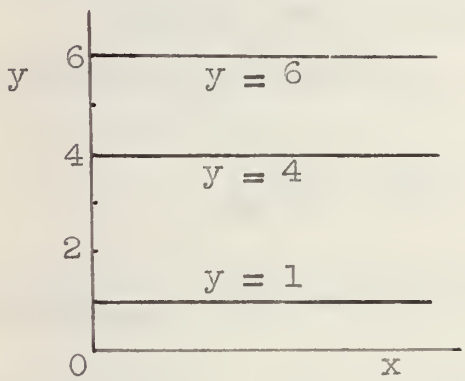


Fig. 21

EXERCISES: Differentiate with respect to x:

(1) $y = 5x^2 - 10x - 7$ (2) $y = \frac{1}{2}x^3 - \frac{1}{4}x^2 - 5$

(3) $y = \frac{3}{4}x^8 - 2$ (4) $y = 12$

Text book: p. 68, nos. 2, 7, 8, 11, 21, 23, 24.

On the basis of the index law: $\sqrt[n]{x^a} = x^{\frac{a}{n}}$, it is possible to use the above formulae to differentiate functions such as: $\sqrt[3]{x^4}$, e.g. $\frac{d}{dx}\sqrt[3]{x^4} = \frac{d}{dx}x^{\frac{4}{3}} = \frac{4}{3}x^{\frac{1}{3}} = \frac{4\sqrt[3]{x}}{3}$

<u>EXERCISES</u>: Differentiate with respect to x:

(1) $y = \sqrt{x^5}$ (2) $y = 12\sqrt[3]{x^7}$ (3) $y = 4\sqrt{x^3}$

(4) $y = \frac{8\sqrt[4]{x^5}}{5}$ (5) $y = \frac{3}{4}\sqrt[3]{x^4}$ (6) $y = \sqrt[n]{x^a}$

Question no. 6 yields a formula which can be used for this type of problem: if $y = \sqrt[n]{x^a}$, $\frac{dy}{dx} = \frac{a}{n} \cdot \sqrt[n]{x^{a-n}}$, but it is only of secondary value.

On the basis of the index law: $\frac{1}{x^n} = x^{-n}$, it is possible to differentiate such functions as $\frac{1}{x^3}$, e.g.

$$\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}x^{-3} = -3x^{-4} = -\frac{3}{x^4} .$$

<u>EXERCISES</u>: Differentiate with respect to x:

(1) $y = \frac{1}{x^2}$ (2) $y = \frac{1}{x^5}$ (3) $y = \frac{5}{x^4}$ (4) $y = \frac{3}{x}$

(5) $y = \frac{1}{2x^3}$ (6) $y = \frac{7}{4x^2}$ (7) $y = \frac{3}{(2x)^3}$ (8) $y = \frac{1}{x^n}$

From the text book: pp. 68-69, nos. 4, 5, 6, 22, 25.

Question no. 8 yields a formula which can be used for this type of problem: if $y = \frac{1}{x^n}$, $\frac{dy}{dx} = \frac{-n}{x^{n+1}}$, but this one too is not as important as some of the others.

<u>NOTE</u>: $x^0 = 1$; $\frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$

<u>EXERCISES</u>: Differentiate with respect to x:

(1) $y = \sqrt{x}$ (2) $y = 5x$ (3) $y = 3\sqrt[3]{x}$

(4) $y = \frac{4}{3\sqrt[4]{x}}$ (5) $y = \frac{12\sqrt[4]{x}}{5}$ (6) $y = \frac{2}{\sqrt[3]{x}}$

From the text book: p. 68, nos. 16, 18, 19, 20.

<u>SUMMARY</u>: The three formulae most useful for differentiating simple functions are:

(1) $\frac{d}{dx}.x^n = n.x^{n-1}$ (2) $\frac{d}{dx}.\ C.x^n = C.n.x^{n-1}$ (3) $\frac{d}{dx}.C = 0$

Two other facts of importance in differentiating are:

(1) Constants occurring as coefficients of x (like C in formula 2 above), "go through" the differentiating unchanged.

(2) Each term in a polynomial expression is differentiated independently of the other terms.

<u>EXERCISES</u>: From the text book: pp. 68-69, nos. 3, 9, 10, 13, 15, 17, 26-33. The student should study the explanation and example from page 68, and then attempt nos. 34-42 on page 69.

If, as x increases, a function (1) increases, it is called an "<u>increasing" function</u>, (2) decreases, it is called a "<u>decreasing" function</u>. Figures 22 and 23 show graphs of increasing and decreasing functions respectively. The slope in fig.22 is positive, and in fig.23, negative, from which the following conclusion may be drawn: $\frac{dy}{dx}$ is (1) <u>positive</u> for an <u>increasing function</u>, and (2) <u>negative</u> for a <u>decreasing function</u>. Of course the same function may be increasing for a certain range of values of x, and decreasing for others. Fig. 24 shows the graph of a function which is increasing between $x = 6\frac{1}{2}$ and $x = 12$, and decreasing from $x = 3$ to $x = 6\frac{1}{2}$. Points such as A, B, and C, where the function

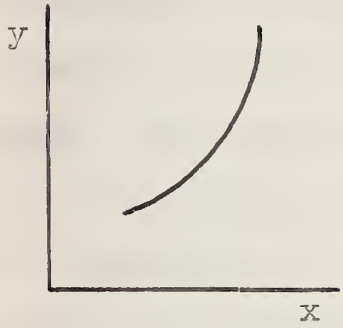


Fig. 22

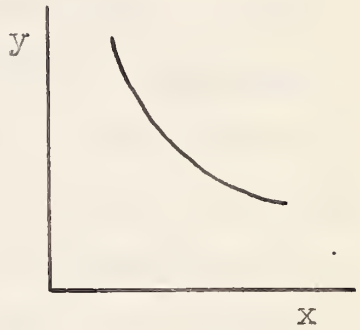


Fig. 23

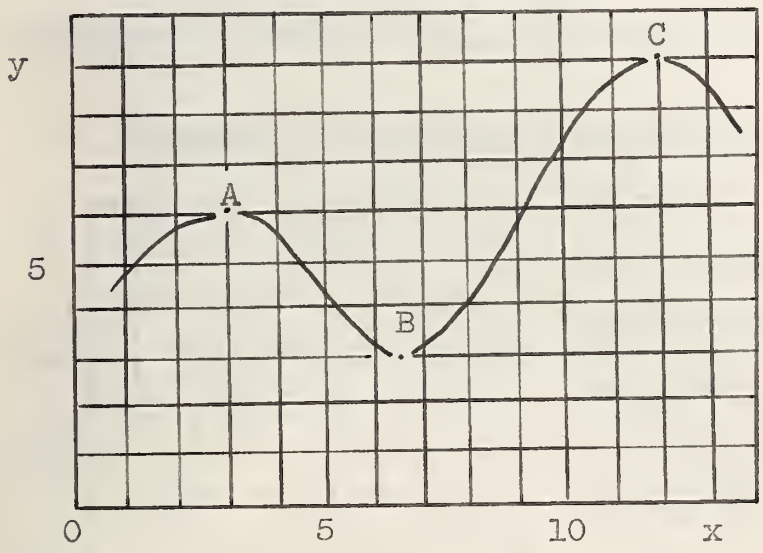


Fig. 24

changes from increasing to decreasing, or vice-versa, are called <u>turning points</u>. There are two kinds of turning points: (1) A is a point where the function changes from increasing to decreasing; it is the highest point in the immediate vicinity, and consequently is called a <u>maximum point</u>. The maximum value of the function in this case is 6. (2) B is a point where the function changes from decreasing to increasing; it is the lowest point in the immediate vicinity, and therefore called a <u>minimum point.</u> The minimum value of the function can be seen to be 3. Notice that the slope of the graph at either type of turning point is zero.

The student should study pp. 70-71 of the text book. The "summary of results" given on pp. 74-75 is also very good.

<u>EXERCISES</u>: From the text book: pp. 71-72, nos. 1 to 5.

<u>Meaning of the second derivative</u>: Fig.25 shows the graph of $y = 33 + 10x - x^2$ as it passes through a maximum turning point (C). Suppose that its slopes where $x = 3$, 4, 5, and 6 (as illustrated by tangents drawn at A, B, C and D) have been determined and recorded in the following table:

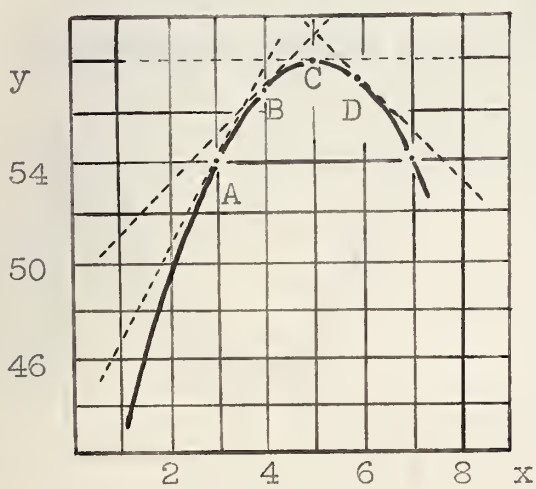


Fig. 25

$\frac{dy}{dx} = 10 - 2x$

x	3	4	5	6
$\frac{dy}{dx}$	4	2	0	-2

It is apparent from both this table, as well as from the tangents themselves, that when the function goes through its maximum point, the slope changes from positive to negative values.

Fig.26 shows a graph based on this table, i.e. a graph of $\frac{dy}{dx} = 10 - 2x$. It should be noted that $10 - 2x$ is a decreasing function having a negative slope. In other words, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ is negative when the original function $(33 + 10x - x^2)$ is at a maximum value.

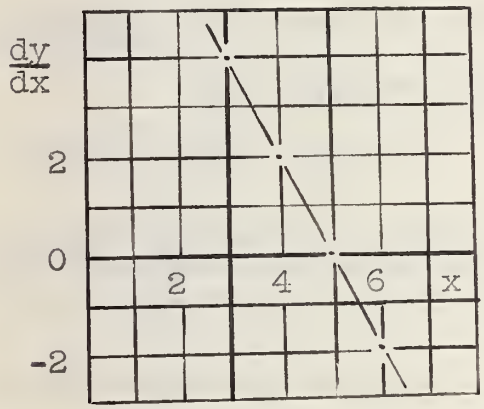


Fig. 26

Fig.27 shows a similar study of the graph of another $f(x)$ as it passes through a minimum turning point.

Fig.28 shows that $\frac{dy}{dx}$ in this case is an increasing function, having a positive slope.

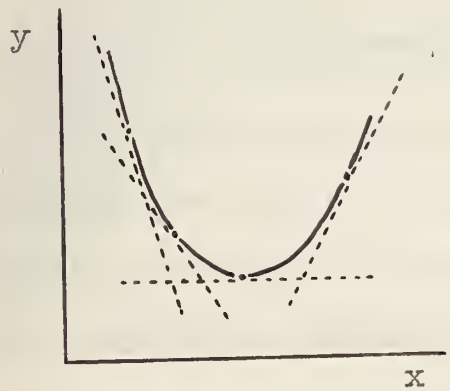


Fig. 27

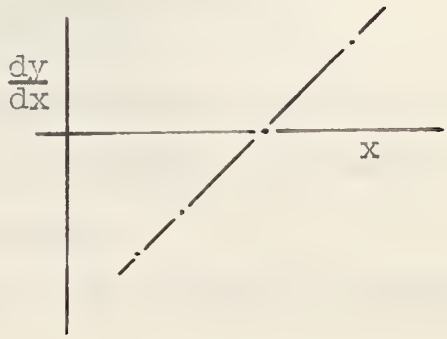


Fig. 28

<u>Conclusion</u>: (1) When a function passes through a maximum point: $\frac{dy}{dx} = 0$ and changes from positive to negative values; $\frac{d^2y}{dx^2}$ is negative.

(2) When a function passes through a minimum point: $\frac{dy}{dx} = 0$ and changes from negative to positive values; $\frac{d^2y}{dx^2}$ is positive.

<u>EXERCISES</u>: (1) From the text book: pp. 73-74, nos. 6 to 12.

(2) Complete the following tables of values:

$y = 2x^3 - 3x^2 - 12x + 7$ $\qquad$ $\frac{dy}{dx} = 6x^2 - 6x - 12$

x	-2	-1	0	1	2	3
y	3	14				

x	-2	-1	0	1	2	3
$\frac{dy}{dx}$	24	0				

On the same set of axes, and with the same scales, draw the graphs of y against x and $\frac{dy}{dx}$ against x (draw the second graph with a dotted or colored line). Explain how these graphs illustrate the conclusions given at the top of this page regarding $\frac{d^2y}{dx^2}$.

Applications of differentiation

Differentiation has many important applications; two of them are studied in grade XII algebra: (1) maxima and minima problems, (2) approximations.

<u>(1) MAXIMA AND MINIMA PROBLEMS</u>: The student is already

familiar with two methods of determining the maximum or minimum value of a function. One of them, the graphical method, is vivid but not always accurate; the other, completing the square, is applicable only to quadratic functions. Differentiation offers a third method which has neither of the limitations of the other two. The basis of this method was given in the conclusion on the preceding page: (1) $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is negative at a maximum value.

(2) $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is positive at a minimum value.

Notice the following example: Find the maximum or minimum values for $y = 2x^3 + 5x^2 - 4x + 1$

$$\frac{dy}{dx} = 6x^2 + 10x - 4$$

Let $6x^2 + 10x - 4 = 0$ $\qquad \frac{d^2y}{dx^2} = 12x + 10$

Then $3x^2 + 5x - 2 = 0$

$(3x - 1)(x + 2) = 0$

And $x = \frac{1}{3}$ or -2

When $x = \frac{1}{3}$, $\frac{dy}{dx} = 0$, and $\frac{d^2y}{dx^2}$ is <u>positive</u> $(+14)$

Therefore, when $x = \frac{1}{3}$, y has a minimum value of

$$f(\tfrac{1}{3}) = \frac{8}{27}.$$

When $x = -2$, $\frac{dy}{dx} = 0$, and $\frac{d^2y}{dx^2}$ is <u>negative</u> (-14)

Therefore, when $x = -2$, y has a maximum value of

$$f(-2) = 13.$$

<u>EXERCISES</u>: From the text book: p. 76, no. 1 (all parts). Find also in each case, the actual maximum or minimum value.

(1) All <u>quadratic functions</u> can be written in the form

$$y = ax^2 + bx + c, \quad \text{and in this case}$$

$$\frac{dy}{dx} = 2ax + b \quad \text{and} \quad \frac{d^2y}{dx^2} = 2a$$

e.g. $y = 2x^2 - 8x + 5$, $\frac{dy}{dx} = 4x - 8$, $\frac{d^2y}{dx^2} = 4$

(a) How can you tell from the value of $\frac{dy}{dx}$ that a quadratic function can have only one turning point ?

(b) What part of the original function tells whether this point is a maximum or a minimum ?

(c) Tell at a glance which of the following have maximum values, and which have minimum values:

$5x^2 + x - 3$; $-2x^2 + x - 4$; $3 + 2x - 4x^2$; $1 - 2x + 3x^2$.

(2) <u>Cubic functions</u> may be written: $y = ax^3$, where a is any number. Then $\frac{dy}{dx} = 3ax^2$ and $\frac{d^2y}{dx^2} = 6ax$

(a) At what value of x is $\frac{dy}{dx} = 0$? How do you know this is neither a max. nor a min. point ?

(b) Fig.29 is a sketch of $y = 2x^3$. Make a sketch for $y = -2x^3$.

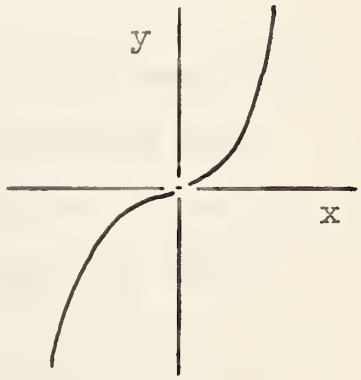


Fig. 29

(3) For the function: $y = \frac{a}{x}$,

$$\frac{dy}{dx} = \frac{-a}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2a}{x^3}$$

(a) How can you tell from the value of $\frac{dy}{dx}$ that this function has no turning points ?

(b) Fig.30 is a sketch of $y = \frac{3}{x}$. Make a sketch for $y = -\frac{3}{x}$.

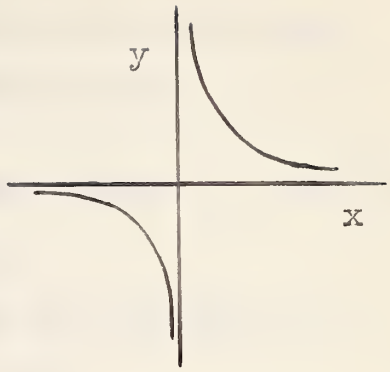


Fig. 30

The student should study the example worked out on pp. 75-76 of the text book. In problems of this type, there are two major parts: (1) Note the measurement for which a maximum or minimum value is to be determined; represent it as a function of <u>one</u> other variable. (2) Use the differentiation method to get the max. or min. value, or the value of x which causes it to have a max. or min. value (which ever is required).

<u>EXERCISES</u>: From the text book: pp. 76-77, nos. 3, 4, 5, 7, 8, 9.

<u>(2) APPROXIMATIONS</u>: $\triangle x$ and $\triangle y$ have already been used to represent corresponding changes in two connected variables, x and y. Consider $y = x^2 - 5$: when $x = 4$, $y = 11$, and when $x = 5$, $y = 20$. In this case, $\triangle x = 1$, and $\triangle y = f(5) - f(4) = 9$.

<u>EXERCISES</u>:

(1) If $y = 3x^2 - 2x - 7$, find $\triangle y$ when $x = 2$ and $\triangle x = 1$.

(2) If $y = \frac{12}{x} + 3$, and x changes from 4 to 6, find $\triangle y$.

(3) From the text book: p. 66, nos. 21 to 25.

A general expression for $\triangle y$ was defined on page 35:

$$\triangle y = f(x + \triangle x) - f(x).$$

This may be used as a formula to obtain an expression for $\triangle y$ in any function. e.g. $y = 3x - x^2$

$$\begin{aligned}\triangle y &= \left[3(x + \triangle x) - (x + \triangle x)^2\right] - (3x - x^2)\\ &= 3x + 3\triangle x - x^2 - 2x.\triangle x - \triangle x^2 - 3x + x^2\\ &= 3\triangle x - 2x.\triangle x - \triangle x^2\end{aligned}$$

Now if x changes from 5 to 3.5,

$$\begin{aligned}\triangle y &= 3(-1.5) - 2(5)(-1.5) - (-1.5)^2\\ &= 8.25\end{aligned}$$

Notice that this process is part of the method used in finding an average gradient, or in getting a gradient by the limit method.

<u>EXERCISES</u>: Obtain an expression for $\triangle y$ in each of the following: (1) $y = x^3$ (2) $y = 7 + 2x - 3x^2$. Using the expressions obtained, find $\triangle y$ when x changes from 2 to 3.

An <u>approximate value for $\triangle y$</u> may be obtained by a shorter method involving differentiation. The formula used is derived on page 78 of the text book: $\triangle y \simeq \frac{dy}{dx}.\triangle x$ (providing $\triangle x$ is <u>small</u>).

After studying page 78, note the following example. In the function $y = 2x^3 - x^2 - 3x - 4$, find $\triangle y$ when x changes from 2 to 2.01.

Exact result

$$\begin{aligned}\triangle y &= f(x + \triangle x) - f(x)\\ &= \left[2(2.01)^3 - (2.01)^2 - 3(2.01) - 4\right] - \left[2(2)^3 - (2)^2 - 3(2) - 4\right]\\ &= 2.171102 - 2\\ &= 0.171102\end{aligned}$$

Approximate result

$$\begin{aligned}\triangle y &\simeq \frac{dy}{dx} \cdot \triangle x\\ &\simeq (6x^2 - 2x - 3) \cdot \triangle x\\ &\simeq \left[6(2)^2 - 2(2) - 3\right](0.01)\\ &\simeq 0.17\end{aligned}$$

It is evident that the approximation method involves much less work, and yields a fairly accurate result--depending on the size of $\triangle x$. Remember the formula: $\triangle y \simeq \frac{dy}{dx} \cdot \triangle x$. The smaller $\triangle x$, the more accurate the approximation.

EXERCISES:

Find an expression showing the approximate value of $\triangle y$:

(1) $y = x^2 - 2x + 7$ (2) $y = x + \frac{1}{x^2}$ (3) $y = 8\sqrt{x}$

(4) Find expressions for the exact and approximate values of $\triangle y$ for $y = 2x^2 - 3x + 1$. What is the error in the approximation when $\triangle x = 0.01$?

(5) When x changes from 1 to 1.02 in $y = 2 + 3x - x^2$ find exact and approximate values for $\triangle y$. What is the error in the approximation ?

(6) Find the approximate value for $\triangle y$ in no.(3) if x changes from 4 to 4.007.

(7) If in no.(1), x changes from 3 to 2.97, find the approximate change in y.

The student should study p. 79 in the text book and then attempt problems from pp.80-81, nos. 1 to 8.

Integration

Integration is the reverse of differentiation; it is related to differentiation in the same way that division is to multiplication.

e.g. $y = 2x^2 + 3x$ is differentiated to give: $\frac{dy}{dx} = 4x + 3$

$\frac{dy}{dx} = 4x + 3$ is integrated to give: $y = 2x^2 + 3x$

Study the example on pp. 85-86 of the text book.

<u>EXERCISES</u>: In what way is the example given above incomplete? From the text book: p. 86, nos. 1 to 10.

It is possible to make up integration formulae. Solving the differential equations

$$\frac{dy}{dx} = x^n , \quad \frac{dy}{dx} = K.x^n , \quad \text{and} \quad \frac{dy}{dx} = K$$

(where n and K are constants) yield the following formulae:

(1) If $\frac{dy}{dx} = x^n$, $y = \frac{x^{n+1}}{n+1} + C$

(2) If $\frac{dy}{dx} = K.x^n$, $y = K.\frac{x^{n+1}}{n+1} + C$

(3) If $\frac{dy}{dx} = K$, $y = K.x + C$

<u>EXERCISES</u>: Solve the following differential equations:

(1) $\frac{dy}{dx} = x^3$ (2) $\frac{dy}{dx} = 3x^5$ (3) $\frac{dy}{dx} = \frac{1}{2}x^2$

(4) $\frac{dy}{dx} = 2x + 3$ (5) $\frac{dy}{dx} = 10x^4 + 4x^3 - 6x^2 + 2x - 7$

(6) $\frac{dy}{dx} = \frac{4x^3}{3} + \frac{6x^2}{5} - \frac{3x}{4} + 2$ (7) $\frac{dy}{dx} = x^6 - x^4 + x^3 - x + 1$

The three formulae stated above may be used for all simple integration exercises provided the index laws are properly employed. e.g.

If $\frac{dy}{dx} = \frac{3}{x^3} = 3x^{-3}$, $y = \frac{3x^{-3+1}}{-3+1} + C = \frac{-3}{2x^2} + C$

<u>EXERCISES</u>: From the text book: p. 87, nos. 11-12

Solve the following differential equations:

(1) $\frac{dy}{dx} = \frac{1}{x^2}$ (2) $\frac{dy}{dx} = \frac{5}{x^4}$ (3) $\frac{dy}{dx} = \frac{1}{2x^2}$

(4) $\frac{dy}{dx} = \frac{3}{x^5} - \frac{2}{5x^3}$ (5) $\frac{dy}{dx} = \frac{7}{x^2} + \frac{4}{x^3} - \frac{6}{x^4}$

(6) $\frac{dy}{dx} = \frac{8}{3x^5} - \frac{1}{2x^2} + \frac{5}{9x^6}$ (7) $\frac{dy}{dx} = \frac{1}{x^3} + \frac{1}{x^4} - \frac{1}{x^5}$

Examples:

(1) Solve: $\frac{dy}{dx} = 5\sqrt[3]{x^2} = 5x^{\frac{2}{3}}$

$$y = \frac{5.x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = 3\sqrt[3]{x^5} + C$$

(2) Solve: $\frac{dy}{dx} = \frac{5}{\sqrt{x^3}} = 5x^{-\frac{3}{2}}$

$$y = \frac{5.x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C = -\frac{10}{\sqrt{x}} + C$$

EXERCISES: Solve the following differential equations:

(1) $\frac{dy}{dx} = \sqrt{x}$

(2) $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^4}}$

(3) $\frac{dy}{dx} = \frac{\sqrt{x^3}}{5}$

(4) $\frac{dy}{dx} = 2\sqrt[3]{x^5} - \frac{2}{\sqrt[3]{x}}$

(5) $\frac{dy}{dx} = \frac{5\sqrt[3]{x^2}}{6} - \frac{11\sqrt[4]{x^7}}{12}$

(6) $\frac{dy}{dx} = \frac{1}{4\sqrt[4]{x^3}} - \frac{2}{3\sqrt[3]{x^5}}$

(7) $\frac{dy}{dx} = \sqrt[3]{x} - \sqrt[5]{x^2}$

(8) $\frac{dy}{dx} = \frac{1}{\sqrt[4]{x}} - \frac{1}{\sqrt[7]{x^4}}$

The value of the arbitrary constant can be determined if a pair of values for x and y are known:

e.g. Given $\frac{dy}{dx} = x + 3$, and $y = 4$ when $x = 3$

Then $y = \frac{1}{2}x^2 + 3x + C$

And putting $x = 3$ and $y = 4$,

$$4 = \tfrac{1}{2}(9) + 9 + C$$

$$C = -\frac{19}{2}$$

Therefore $y = \tfrac{1}{2}x^2 + 3x - \frac{19}{2}$

EXERCISES: Text book, p. 87, nos. 13 to 21.

There is another way of writing an integration problem: e.g. $\frac{dy}{dx} = x^2 - 3$ is often written $y = \int(x^2 - 3)dx$. $\int$ is the integration sign, and the second statement above is read--"y equals the integral of $x^2 - 3$". The "dx" means with respect to x. $\int \ldots dx$ may also be considered a sign of operation meaning--"integrate with respect to x".

The integration formulae given above may be stated in the optional form:

(1) $\int x^n . dx = \frac{x^{n+1}}{n+1} + C$

(2) $\int K . x^n . dx = \frac{K . x^{n+1}}{n+1} + C$

(3) $\int K . dx = K . x + C$

<u>REVIEW EXERCISES</u>: Integrate:

(1) $\int(\tfrac{1}{2}x^2 - 3x + 4)dx$

(2) $\int(\frac{2}{x^3} - \frac{3}{5x^4})dx$

(3) $\int(\frac{7\sqrt[4]{x^3}}{4} - \frac{2}{3\sqrt[3]{x^2}})dx$

Applications of Integration

Integration is used for two purposes in grade XII algebra: (1) for finding areas, and (2) for finding volumes.

(1) <u>AREAS:</u> Example II on pages 89-90 of the text book shows the derivation of a formula for obtaining an area between the graph of $y = \frac{x^2}{5} + 1$ and the x-axis. Note that ON = x, NP = y, CONP = A (required area), and that the corresponding changes in these measurements are NM = $\triangle x$ (δ is another form of delta), RQ = $\triangle y$, PNMQ = $\triangle A$. The example proves that $A = \int(\frac{x^2}{5} + 1)dx$.

The formula $A = \frac{x^3}{15} + x$ may be used to find the area (shown in fig.31) enclosed by the graph of $y = \frac{x^2}{5} + 1$, the y-axis. the x-axis, and the ordinate drawn at any point on the x-axis. If OB = 5, then $A = \frac{5^3}{15} + 5 = 13\frac{1}{3}$

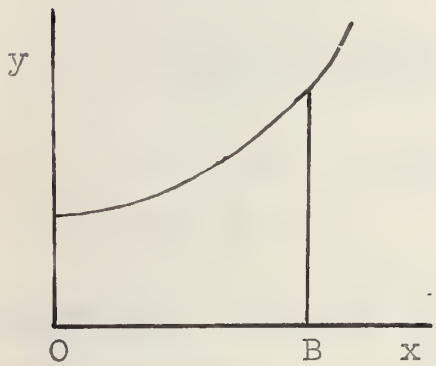


Fig. 31

<u>NOTE</u>: one "unit" of area is one unit (on the x-scale) wide and one unit (on the y-scale) high.

<u>EXERCISES</u>: Write down the formula used to find the area under the graph of each of the following functions:

(1) $y = 2x + 7$ (2) $y = 4x^3 + 5$

Use each formula to find the area bounded by the axes, the graph of the function, and the ordinate at $x = 2$.

Example III on pages 90-91 should be studied carefully.

<u>Note the very important area formula:</u> $\int_a^b f(x)dx$

It represents the area (shaded in fig.32) enclosed by the graph of $y = f(x)$, the x-axis, and the ordinates at $x = a$ and $x = b$. A definite integral, like the above formula, is evaluated as follows:

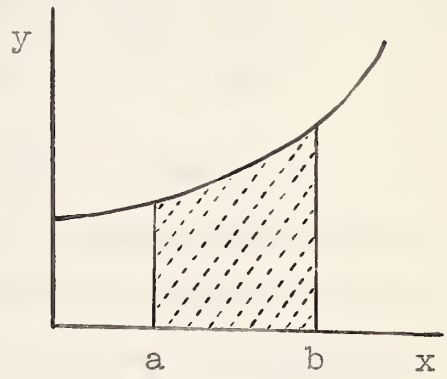


Fig. 32

$$\int_{-1}^{3}(22 - 5x - 3x^2)dx$$

$$= \left[22x - \frac{5x^2}{2} - x^3\right]_{x=-1}^{x=3}$$

$$= \left[22(3) - \frac{5(3)^2}{2} - (3)^3\right] - \left[22(-1) - \frac{5(-1)^2}{2} - (-1)^3\right]$$

$$= (16\tfrac{1}{2}) - (-23\tfrac{1}{2})$$

$$= 40$$

<u>EXERCISES</u>: Evaluate the following definite integrals. Sketch a graph in each case to indicate the area being found.

(1) $\int_2^5 x^3.dx$ (2) $\int_{-3}^{1}(x^2 + 1)dx$

(3) $\int_1^4 \frac{1}{x^2}.dx$ (4) $\int_4^9 \sqrt{x}.dx$

From the text book: pp. 92-93, nos. 1 to 7.

<u>Example</u>: Find the area bounded by the graph of $y = x^2 - 5x$, the x-axis, and the ordinate at $x = 3$. The area is illustrated in fig.33, and is calculated by evaluating $\int_0^3 (x^2 - 5x)dx = -13\frac{1}{2}$.

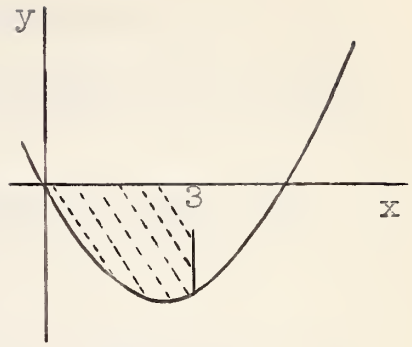


Fig. 33

The negative sign of the area, indicates that it is below the x-axis. Because of this, care must be exercised in finding areas like the shaded portion in fig.34.

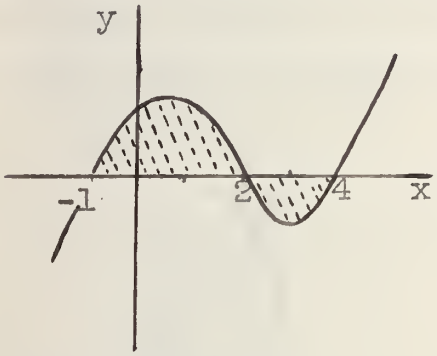


Fig. 34

The method used is to integrate between $x = -1$ and $x = 2$ to get the first part of the area, and then integrate between $x = 2$ and $x = 4$ to get the area of the other loop. The sum of these (neglecting the negative sign of the second) will give the total area required. Integrating between $x = -1$ and $x = 4$ would give the difference of the areas of the two loops.

<u>EXERCISES</u>:

(1) Find the area between the graph of $(x - 2)(5 - x)$ and the x-axis.

(2) Find the area enclosed by the graph of $(x + 1)(x + 3)$ and the x-axis.

(3) Find the area bounded by the graph of $x(x - 4)$ and the x-axis.

(4) From the text book: page 94, number 12.

(5) Find the area enclosed by the graphs of $y = \frac{1}{4}x^2$ and $y = 2\sqrt{x}$

(6) Find the area between the graphs of $2x^2 - 6x$ and $3x - x^2$.

(2) <u>VOLUMES</u>: The derivation of the volume formula is very similar to the derivation of the area formula. In the text book, exercise 8 on page 93 is used to introduce the method of getting volumes by integration. The example given here is this exercise worked out. Figure 35 shows the solid formed when the graph of $y = x^2$ is rotated around the x-axis. If $ON = x$, $NM = \triangle x$, and the required volume between O and the cross-section at N is V, then: $NP = y$, $MQ = y + \triangle y$, and the volume between the cross-section at N and the cross-section at M is $\triangle V$. The area of the circular cross-section at N is πy^2, and at M is $\pi(y + \triangle y)^2$. Then $\pi y^2 . \triangle x$ is the volume of the small cylinder between N and M (partly shown in dotted lines), and $\pi(y + \triangle y)^2 . \triangle x$ is the volume of the large cylinder between N and M (also partly shown in dotted lines). Evidently: $\pi y^2 . \triangle x < \triangle V < \pi(y + \triangle y)^2 . \triangle x$

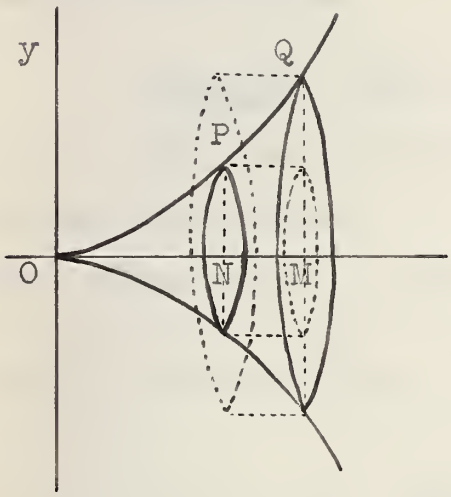


Fig. 35

or: $\pi y^2 < \frac{\Delta V}{\Delta x} < \pi(y + \Delta y)^2$. When $\Delta x \to 0$, this statement yields: $\frac{dV}{dx} = \pi y^2$ from which $V = \int \pi(x^2)^2 . dx = \pi \frac{x^5}{5} + C$.

Since volume = 0 when $x = 0$, the value of C must be 0. Therefore the volume between 0 and the cross-section at N is $\pi \frac{x^5}{5}$.

<u>A general volume formula is</u>: $V = \int_a^b \pi \left[f(x)\right]^2 . dx$

This is the volume (between cross-sections at $x = a$ and $x = b$) generated when the graph of a function of x is rotated around the x-axis.

<u>Example</u>: Find the volume between cross-sections at $x = 1$ and $x = 2$ when the graph of $y = x^2 + 2$ is rotated around the x-axis.

The volume--shown in figure 36-- is given by:

$$V = \int_1^2 \pi(x^2 + 2)^2 . dx$$

$$= \int_1^2 \pi(x^4 + 4x^2 + 4) dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x\right]_{x=1}^{x=2}$$

$$= 25.06\pi - 5.53\pi = 19.53\pi$$

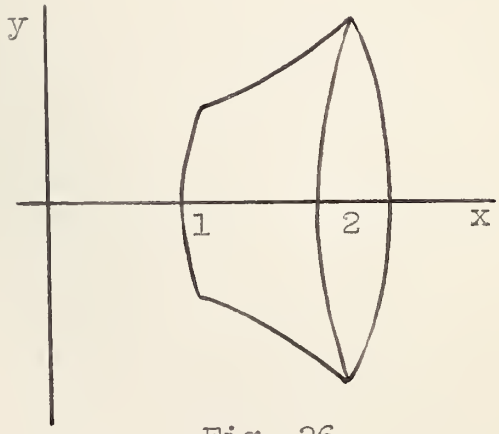


Fig. 36

<u>EXERCISES</u>: Find the volume generated when the graph of each of the following functions is rotated around the x-axis. Make a sketch to illustrate each:

(1) $y = \frac{1}{x}$, between cross-sections at $x = 1$ and $x = 5$.

(2) $y = \sqrt{x}$, between cross-sections at $x = 4$ and $x = 9$.

(3) From the text book: pp. 93-94, nos. 8(parts 7-8-9), 9, 10.

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